Multifractals and Wavelets in Turbulence
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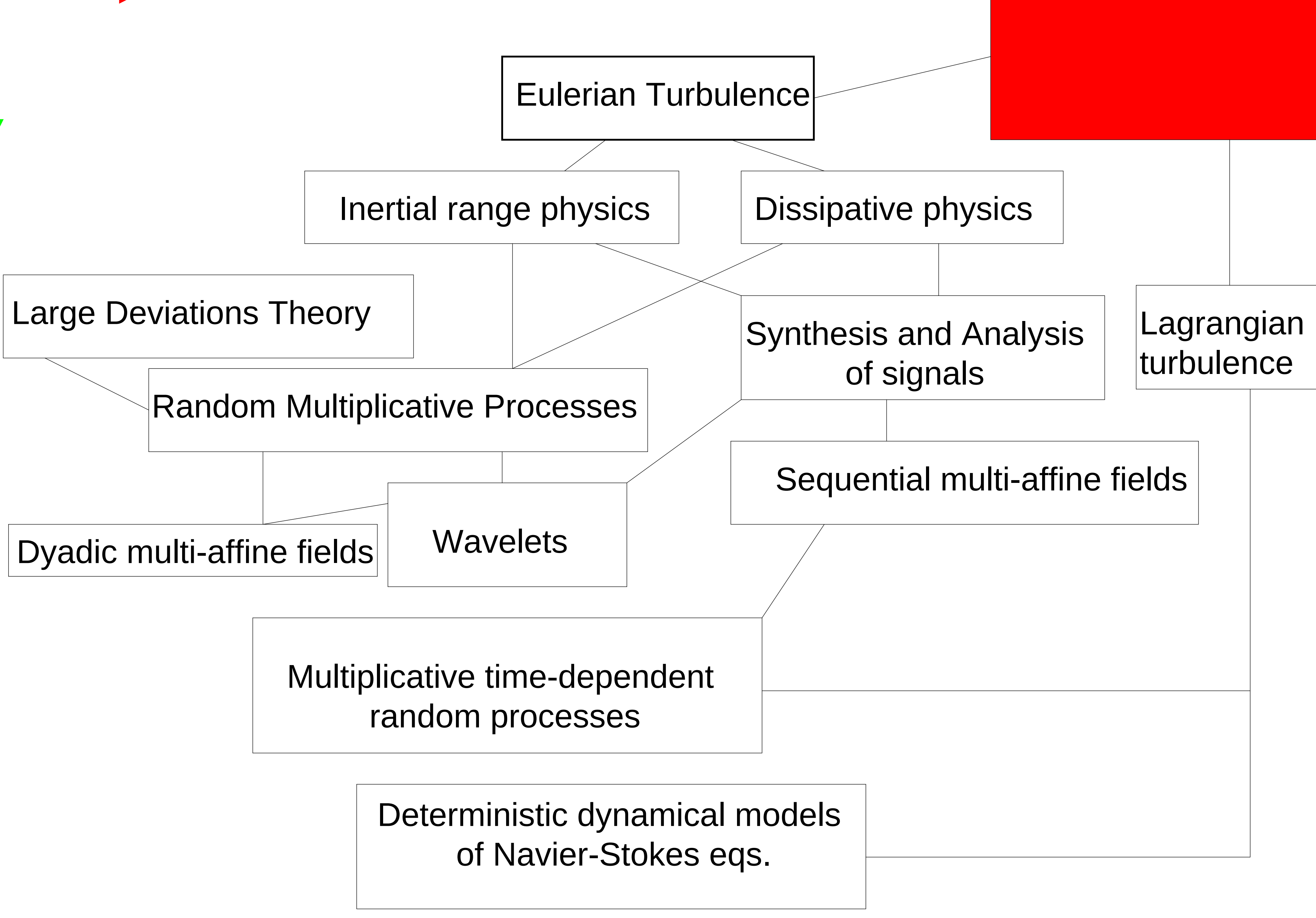
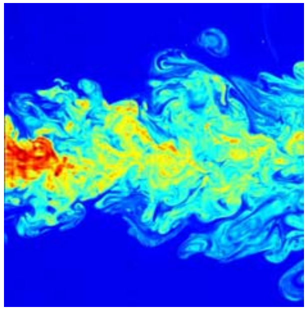


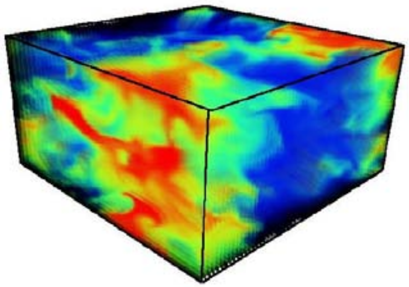
Eulerian Turbulence
Inertial range physics
Dissipative physics
Large Deviations Theory
Synthesis and Analysis of signals
Lagrangian turbulence
Random Multiplicative Processes
Sequential multi-affine fields
Dyadic multi-affine fields
Wavelets
Multiplicative time-dependent random processes
Deterministic dynamical models of Navier-Stokes eqs.

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial} \boldsymbol{v} = -\boldsymbol{\partial} P + \nu \partial^2 \boldsymbol{v} + \boldsymbol{f} \\ \boldsymbol{\partial} \cdot \boldsymbol{v} = 0 \\ + \text{ boundary conditions} \end{cases}$$

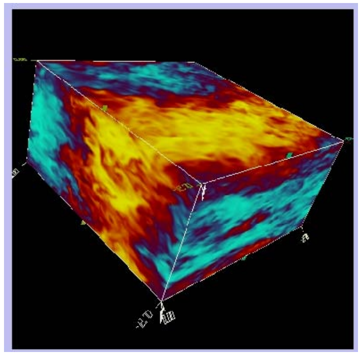
Kinematics + Dissipation are invariant under Rotation+Translation



Turbulent jet



3d Convective Cell



Shear Flow

Small-scale statistics: are there universal properties?

Ratio between non-universal/universal components at different scales

Physical Complexity

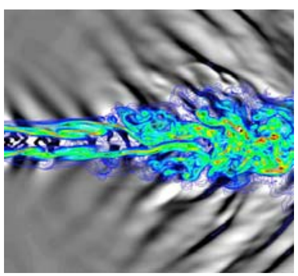
$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial v} = -\boldsymbol{\partial} P + \nu \partial^2 \boldsymbol{v} \\ \partial_t \bar{v} + \bar{v} \cdot \partial \bar{v} = -\partial \bar{P} + \frac{1}{Re} \partial^2 \bar{v} \end{cases}$$

$Re : \frac{U_0 L_0}{\nu}$ Reynolds number ~ (Non-Linear)/(Linear terms)

•Fully Developed Turbulence: $Re \to \infty$

Strongly out-of-equilibrium non-perturbative system

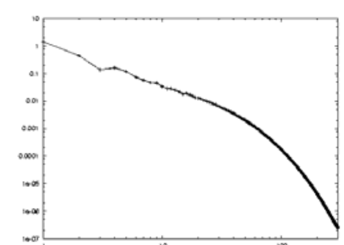
$$\lim_{\nu \to 0} \ \epsilon = \nu \langle (\boldsymbol{\partial v})^2 \rangle \to const.$$



Many-body problem

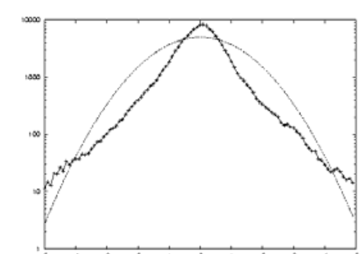
•Power laws:

$$\#_{dof} = (\frac{k_0}{k_\eta})^3 \propto Re^{9/4}$$



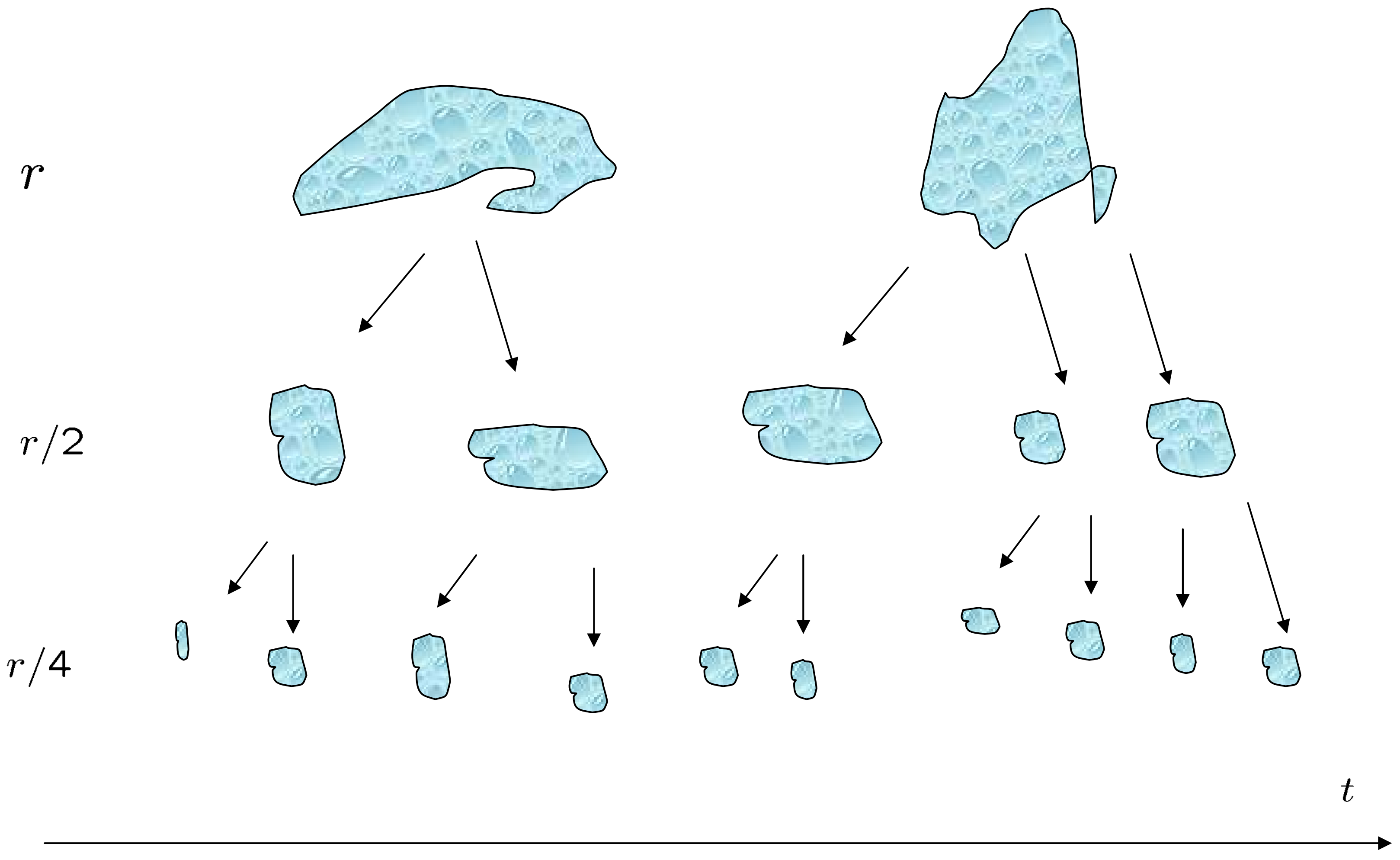
Energy spectrum

•Small-scales PDF strongly non-Gaussian



acceleration

spatio-temporal Richardson cascade



$$\delta_{\boldsymbol{r}} v^{\alpha}(t) = v^{\alpha}(\boldsymbol{x}, t) - v^{\alpha}(\boldsymbol{x} + \boldsymbol{r}, t)$$

$$S_n^{\bar{\alpha}}(\bar{\boldsymbol{r}}, \bar{t}) = \langle \delta_{\boldsymbol{r}_1} v^{\alpha_1}(t_1) \cdots \delta_{\boldsymbol{r}_n} v^{\alpha_n}(t_n) \rangle$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\partial}\boldsymbol{v} = -\boldsymbol{\partial}P + \cancel{\nu\partial^2\boldsymbol{v}} + \cancel{\boldsymbol{f}}$$

$$\boxed{\begin{aligned} \boldsymbol{v}' &\to \lambda^h \boldsymbol{v} \\ \boldsymbol{x}' &\to \lambda \boldsymbol{x} \\ t' &\to \lambda^{1-h} t \end{aligned}} \longrightarrow \forall h$$

<u>Scaling invariance in the Inertial Range</u>

Third order longitudinal structure functions:

$$S_3(r) = \langle (\hat{\boldsymbol{r}}\cdot\delta_{\boldsymbol{r}}\boldsymbol{v})^3\rangle$$

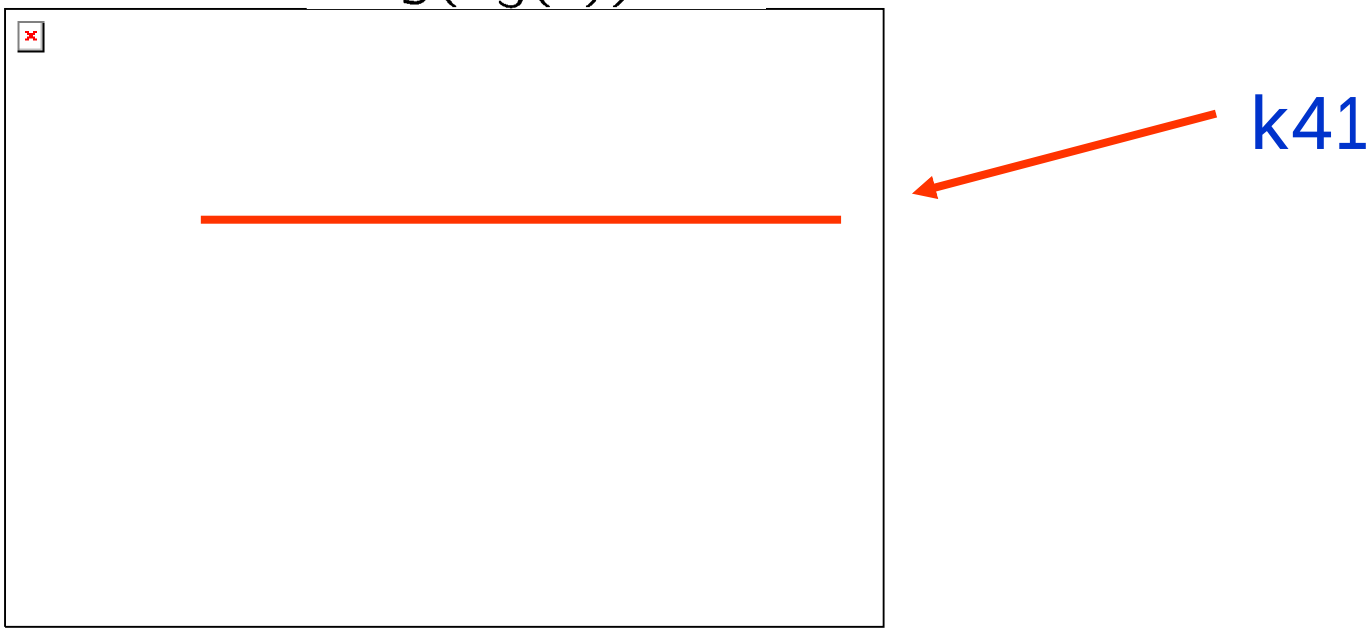
$$\boxed{S_3(r) = -\frac{4}{5}\epsilon r + \cancel{6\nu\frac{dS_2(r)}{dr}} + \cancel{O(r^3)}} \rightarrow h = \frac{1}{3}$$

EXACT FROM NAVIER-STOKES EQS.

Kolmogorov 1941

$$S_p(r) = \langle (\hat{\boldsymbol{r}} \cdot \delta_{\boldsymbol{r}} \boldsymbol{v})^p \rangle \sim \epsilon^{p/3} r^{p/3}$$

Logarithmic local slopes $$\frac{d \log(S_p(r))}{d \log(S_3(r))} = \frac{p}{3}$$



Local slope of 6th order structure function in the isotropic sector, at changing Reynolds and large scale set-up.

Exp.	Configuration	A	η	R_λ	u'/U (%)	l_w/η	f_a/f_η	Ref.
1	swirling flow	10 cm	2.5-50 μm	200-5000	20-40	0.1-3	0.5-5	[2]
2a	jet	20 cm	0.28 mm	428	26	2.5	7	[3]
2b	wind tunnel	10 cm	0.35 mm	3050	7	1.2	3	
3	jet	1 cm	7 μm	580	25	3	7	[4]
4a	cylinder	6-10 cm	0.2-0.5 mm	100-300	15	1-2.5	7	[5]
4b	jet	10 cm	0.1 mm	800	30	5	7	
5a	jet	7.5 cm	0.095 mm	810	16	2	1	[6]
5b	grid	17 cm	0.19 mm	530	8	1	1	
6	jet	4-8 cm	22-48 μm	240-330	20-25	0.6-1.3	–	[7]
7	grid	4 mm-1 cm	100-250 μm	35-110	1.5-8	3-10	1-3	[8]

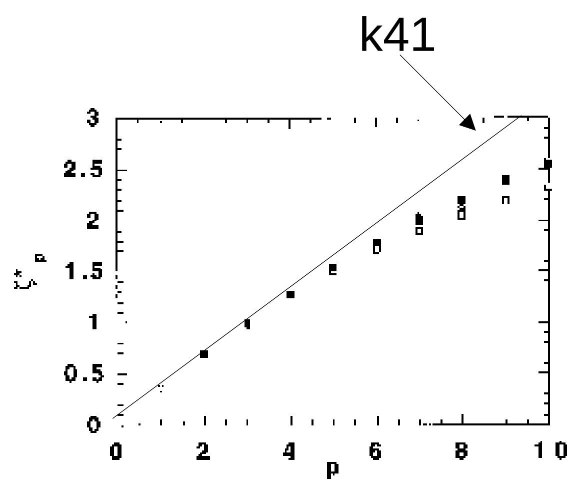


Fig. 3. – Evolution, with p, of the structure function exponents ζ_p^*, for different experiments: □ exp. 1 (the exponents are found independent of R_λ), × exp. 2a, ● exp. 2b, ♦ exp. 3, ■ exp. 5a, ▲ exp. 5b, ○ exp. 6, + exp. 7.

Simple Eulerian multifractal formalism

“local” scaling invariance

$$\begin{cases} \delta_r v \sim v_L (\frac{r}{L})^{h(x)} \\ \langle (\delta_r v)^p \rangle_x \sim \int dh\, (\frac{r}{L})^{hp} P_r(h) \end{cases}$$

$D(h)$: Fractal dimension of the set $\{\boldsymbol{x} : \delta_r v \sim r^{h(\boldsymbol{x})}\}$

$$P_r(h) \sim r^{3-D(h)}$$

$$\begin{cases} \langle (\delta_r v)^p \rangle \sim \int dh\, (\frac{r}{L})^{hp} (\frac{r}{L})^{3-D(h)} \sim r^{\zeta(p)} \\ \zeta(p) = \min_h (hp + 3 - D(h)) \end{cases}$$

What about PDF?

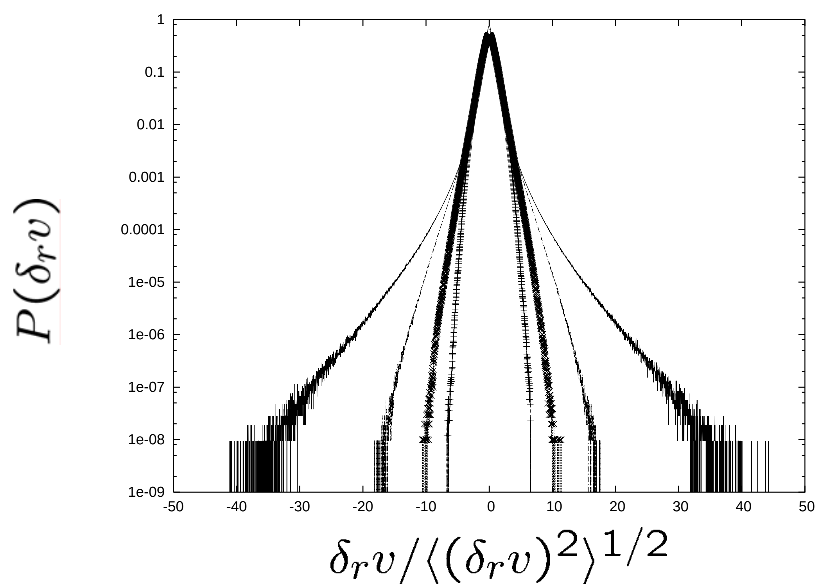
$$\delta_r v \sim v_L (\tfrac{r}{L})^h$$

Experimental results tell us PDF at large scale is close to Gaussian

$$P(v_L) \sim \mathrm{exp}(\tfrac{v_L^2}{2})$$

$$P(\delta_r v) \sim \int dh\, (\tfrac{r}{L})^{3-h-D(h)}\, \mathrm{exp}(-\frac{(\delta_r v)^2}{2(r/L)^{2h}})$$

Superposition of Gaussians with different width:

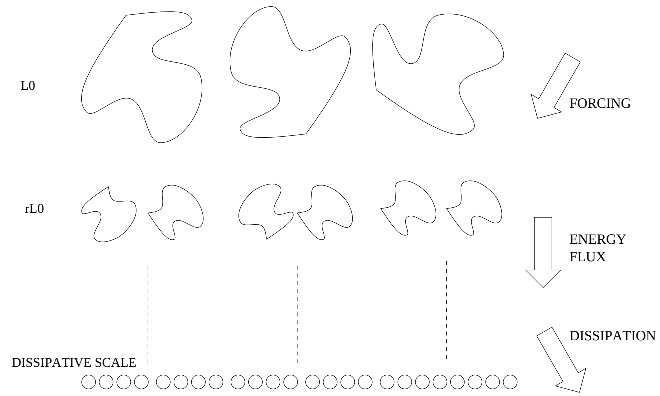


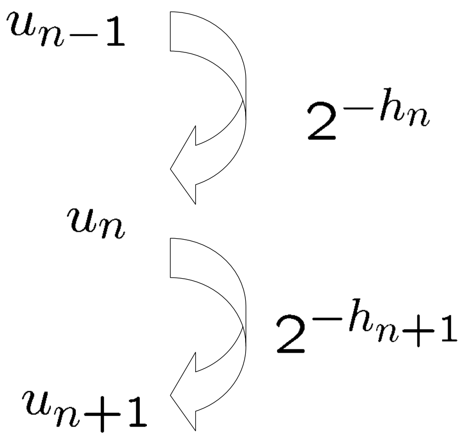
How to derive D(h) from the equation of motion?

Physical intuition of D(h): the result of a random energy cascade

$$r_n = 2^{-n} L$$

$$u_n = \delta_{r_n} v$$





$$u_n = 2^{-h_n} u_{n-1}$$

Large deviation theory

$$u_n = (\Pi_{i=1}^{n} 2^{-h_i}) u_0 \equiv 2^{-n(\frac{1}{n}\sum_{i=1}^{n} h_i)} u_0$$

$$P(h = \tfrac{1}{n}\sum_{i=1}^{n} h_i)) \sim 2^{-nS(h)}$$

$$\langle u_n^p \rangle \sim u_0^p \int dh (\tfrac{r_n}{L})^{hp+S(h)}$$

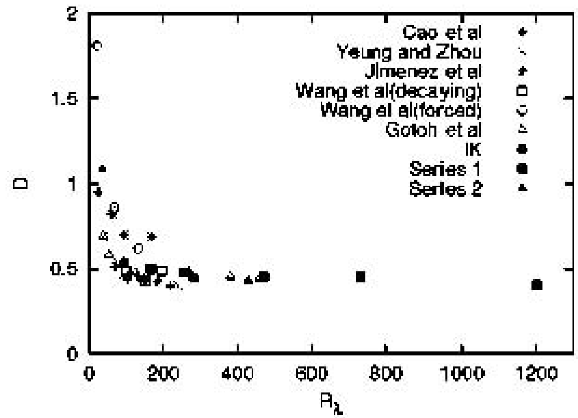
! Scaling is recovered in a statistical sense, no local scaling properties !

Looking for other physical observable: the physics of dissipation

Energy dissipation is Reynolds independent:
<u>Dissipative anomaly</u>

$$\lim_{Re\to\infty} \equiv \lim_{\nu\to 0}$$

$$\epsilon = \nu\langle(\boldsymbol{\partial v})^2\rangle \to const.$$



How to derive the statistics of gradients within the multifractal formalism?

$$Re(r) = \frac{r\,\delta_r u}{\nu}$$

$$\boldsymbol{v}\cdot\boldsymbol{\partial v} \sim \nu\partial^2\boldsymbol{v} \longrightarrow Re(\eta) \sim O(1) \longrightarrow \frac{\eta\,\delta_\eta u}{\nu} \sim O(1)$$

$$\delta_\eta v \sim v_L\left(\frac{\eta}{L}\right)^h \longrightarrow \eta^{1-h} \sim \nu L^h/v_L$$

★ <u>Dissipative scale fluctuates</u> ★

2 consequences:

- Intermediate dissipative range

$$\eta_{min} < r < \eta_{max}$$

$$\langle(\delta_r v)^p\rangle \sim \int_{h_{min}(r)} dh\,(\tfrac{r}{L})^{hp}(\tfrac{r}{L})^{3-D(h)} \sim r^{\zeta(p,r)}$$

- Statistics of gradients highly non trivial

$$s = \frac{\delta_\eta v}{\eta} \qquad\qquad s = v_L \eta^{h-1}/L^h$$

$$P(s) = \int dh\, dv_L P_\eta(h) P(v_L)$$

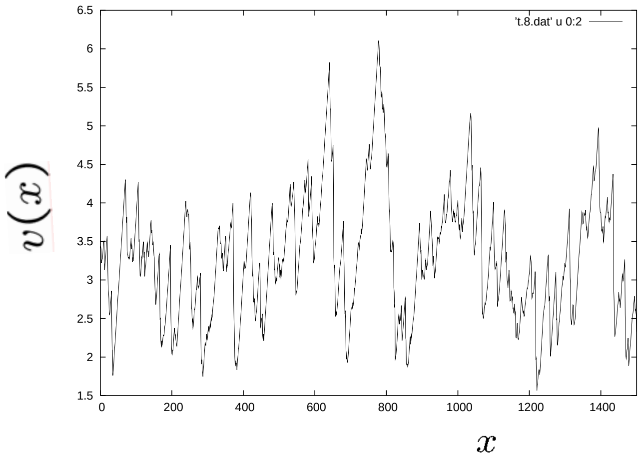
$$P(s) = \int dh(\tfrac{\nu}{s})^{y(h)} \exp(-\frac{\nu^{1-h} s^{1+h}}{2\langle v_L^2\rangle}) \qquad y(h) = \frac{4-[h+D(h)]}{2}$$

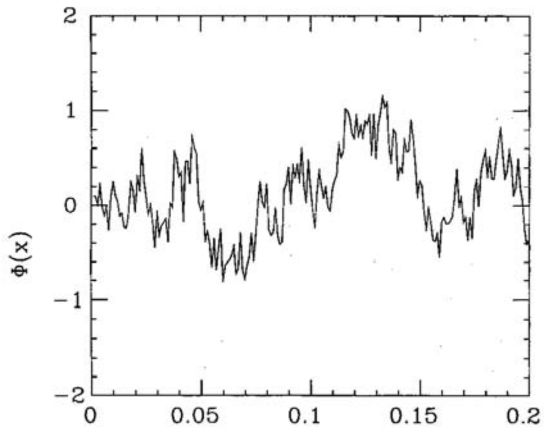
★ $\frac{s}{\langle s^2\rangle^{1/2}} > 1$ ★

$$\boxed{\langle s^p\rangle \sim Re^{\zeta(p)}}$$

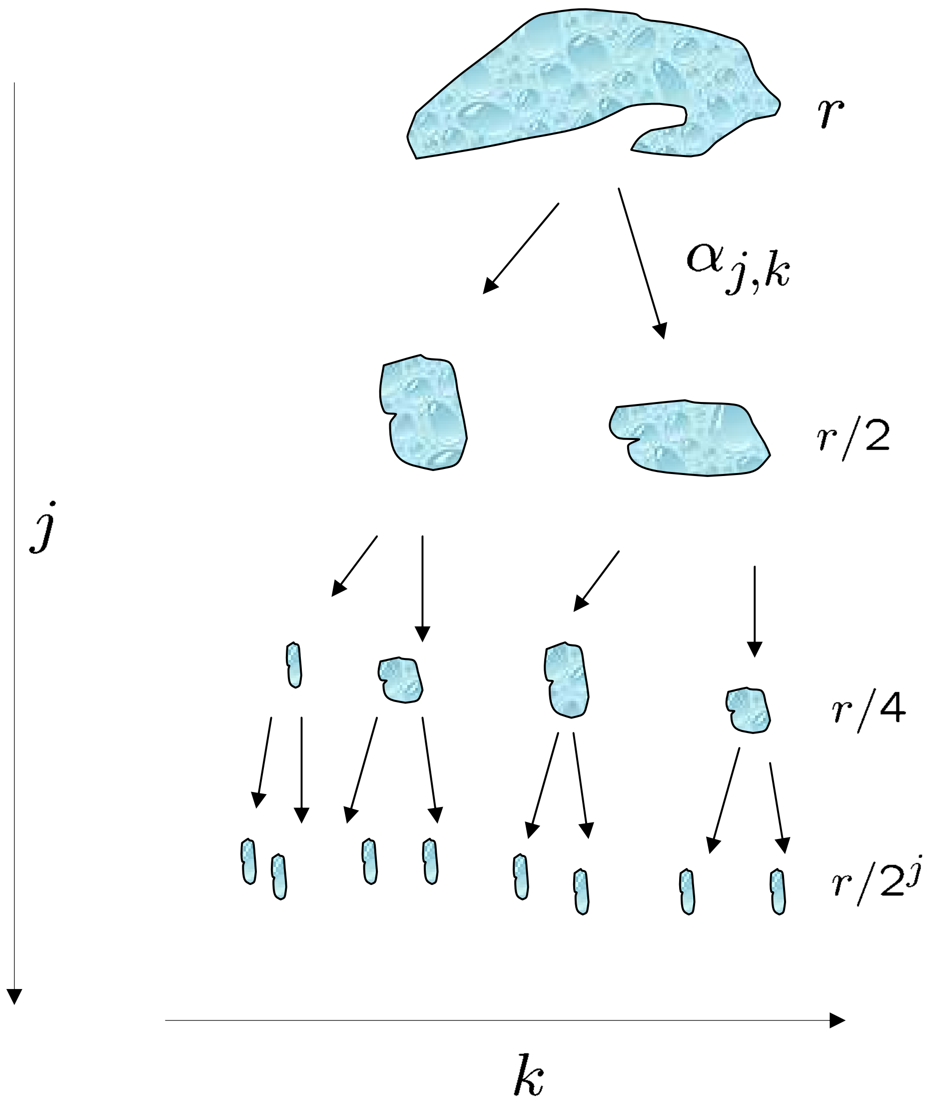
Synthesis & Analysis

- How to build a multiaffine field with prescribed scaling laws
- How to distinguish synthetic and real fields





<u>Richardson cascade: random multiplicative process</u>



$$v(x) = \Sigma_{j=0}^{N-1} \Sigma_{k=0}^{2^j-1} \alpha_{j,k} \psi_{j,k}(x)$$

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$$

$$\langle |v(x+r) - v(x)|^p \rangle \sim r^{\xi(p)}$$

$$\xi(p) = -\frac{1}{2} p - \log_2(\langle A^p \rangle)$$

$$\alpha_{0,0} = 1$$

$$\alpha_{1,0} = A_{1,0}\alpha_{0,0} \qquad \alpha_{1,1} = A_{1,1}\alpha_{0,0}$$

$$\alpha_{2,0} = A_{2,0}\alpha_{1,0} \qquad \alpha_{2,1} = A_{2,1}\alpha_{1,0}$$

Multiplicative uncorrelated structure

$$\langle|\alpha_{j,k}|^p\rangle = \langle A^p\rangle\langle|\alpha_{j-1,k}|^p\rangle = 2^{j\log_2(\langle A^p\rangle)}\langle|\alpha_{0,0}|^p\rangle$$

$$\boxed{\begin{gathered}\langle|v(x+r)-v(x)|^p\rangle \sim r^{\xi(p)} \\ \xi(p) = -\frac{1}{2}p - \log_2(\langle A^p\rangle)\end{gathered}}$$

$$S_2(r) = \langle (v(x+r) - v(x))^2 \rangle$$

$$S_2(r) = \langle \Sigma_{j,k} (\alpha_{j,k} 2^{j/2} (\psi(2^j x + 2^j r - k) - \psi(2^j x - k)))^2 \rangle$$

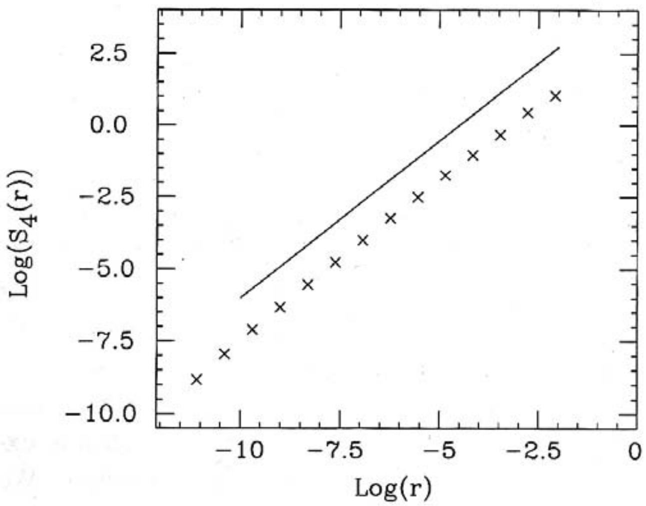
+ Spatial Ergodicity

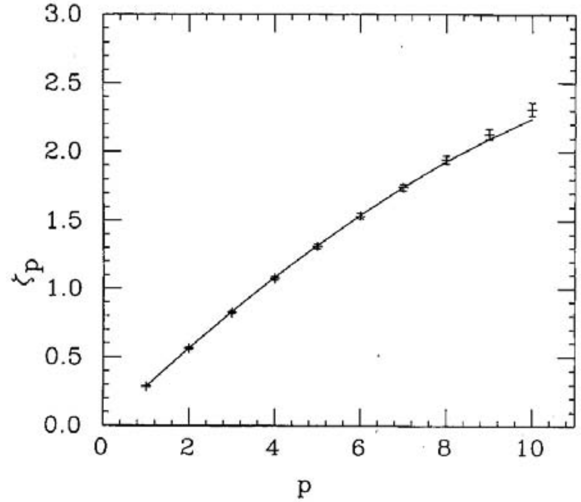
$$S_2(r) = \Sigma_{j,k} 2^j \langle \alpha_{j,k}^2 \rangle \langle (\psi(2^j x + 2^j r - k) - \psi(2^j x - k))^2 \rangle$$

$$G_2(r) = \int dx (\psi(x+r) - \psi(x))^2 \qquad S_2(r) = \Sigma_j 2^j \langle \alpha_{j,k}^2 \rangle G_2(2^j r)$$

$$S_2(2r) = \Sigma_j 2^j \langle \alpha_{j,k}^2 \rangle G_2(2^{j+1} r) = \Sigma_j 2^{j(1+\log_2(\langle A^2 \rangle))} G_2(2^{j+1} r)$$

$$S_2(2r) = 2^{-(1+\log_2(\langle A^2 \rangle))} \Sigma_j 2^{(j+1)(\log_2(\langle A^2 \rangle + 1))} G_2(2^{j+1} r) = 2^{-(1+\log_2(\langle A^2 \rangle))} S(r)$$



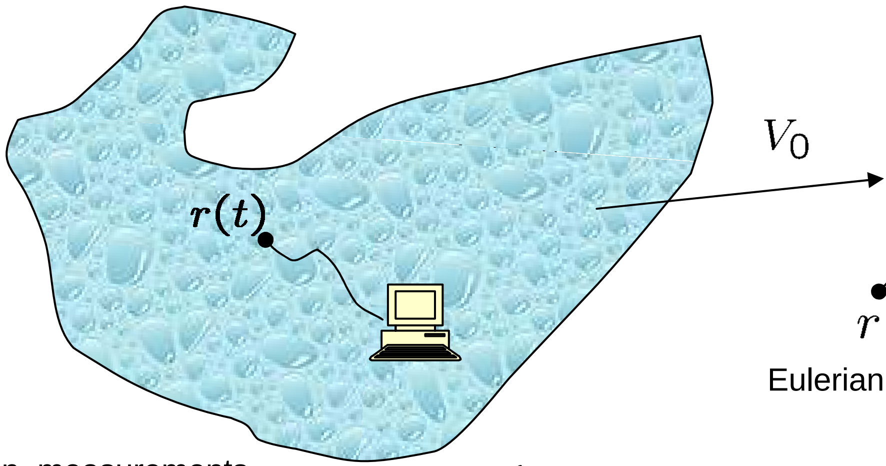


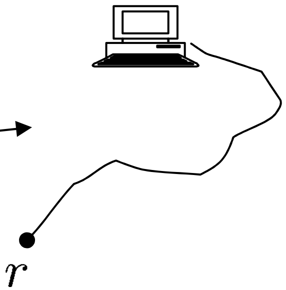
$$\langle |v(x+r) - v(x)|^p \rangle \sim r^{\xi(p)}$$

$$\xi(p) = -\frac{1}{2}p - \log_2(\langle A^p \rangle)$$

- Physics of dissipation easily implemented by changing distributions of multipliers
- What about 2d and 3d fields: possible theoretically, much more hard numerically
- What about divergence-less fields: same as before
- What about temporal and spatial scaling? Where are the Navier-Stokes eqs?

Wavelets, Multiplicaitive processes, Diadic structure and time properties

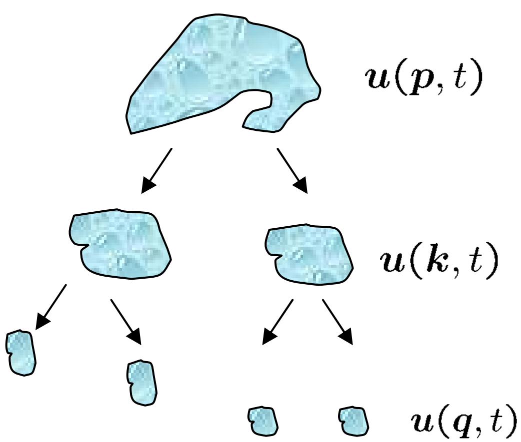




Eulerian measurements $v(r,t)$

Lagrangian measurements
$v(r(t),t)$

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\partial}\boldsymbol{v} = -\boldsymbol{\partial}P + \nu\partial^2\boldsymbol{v} + \boldsymbol{f} \\ \dot{\mathbf{r}}(t) = \boldsymbol{v}(\mathbf{r}(t),t) \end{cases}$$



$(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0)$

Constraint from the equation of motion

$$\partial_t \boldsymbol{u}(\boldsymbol{k}) \sim (\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{p}))\boldsymbol{u}(\boldsymbol{q})$$

$$\tau^{-1}(k) \sim k u(k,t)$$

Fluctuating local eddy-turn-over time

Simple multifractal formalism
Eulerian vs Lagrangian

Eulerian:

$$\begin{cases} \delta_r v \sim r^h \\ P_r(h) \sim r^{3-D(h)} \end{cases}$$

$$\langle (\delta_r v)^p \rangle \sim \int dh\, r^{hp} r^{3-D(h)} \sim r^{\zeta_E(p)}$$

$$\zeta_E(p) = \min_h (hp + 3 - D(h))$$

Lagrangian

$$\begin{cases} \delta_\tau v \equiv v(r(t+\tau), t+\tau) - v(r(t), t) \sim \tau^{\frac{h}{1-h}} \\ \tau^{-1} \sim \delta_r v / r \sim r^{h-1} \end{cases}$$

$$\langle (\delta_\tau v)^p \rangle \sim \int dh\, \tau^{\frac{hp+3-D(h)}{1-h}} \sim \tau^{\zeta_L(p)}$$

$$\zeta_L(p) = \min_h \left(\frac{hp+3-D(h)}{1-h}\right)$$

$$S_n^{\bar{\alpha}}(\bar{r}, \bar{t}) = \langle \delta_{r_1} v^{\alpha_1}(t_1) \cdots \delta_{r_n} v^{\alpha_n}(t_n) \rangle$$ Multi-particle

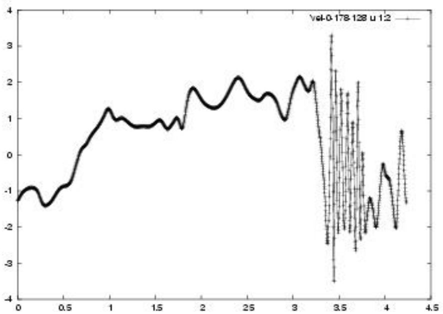
$$\left.\begin{aligned} v(t) &= \sum_n u_n(t) \\ u_n(t) &= x_1(t) x_2(t) \cdots x_n(t) \\ dx_j(t) &= -\frac{1}{\tau_j}\frac{dV}{dx_j}dt + \sqrt{2/\tau_j}\, dW_j \end{aligned}\right\} \longrightarrow \langle (\delta_\tau v)^p \rangle \sim \tau^{\zeta_L(p)}$$

Needing for “sequential” multiaffine functions/measures

High resolution for following particles

Typical velocity and acceleration

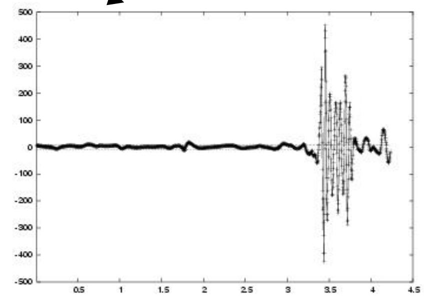
$v_x(t)$



t

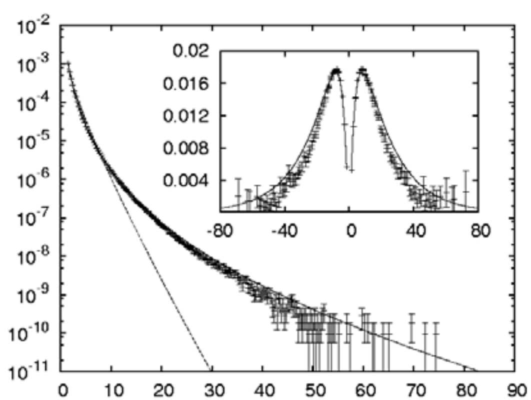
$$\dot{\mathbf{r}}(t) = \boldsymbol{v}(\mathbf{r}(t), t)$$

$a_x(t)$



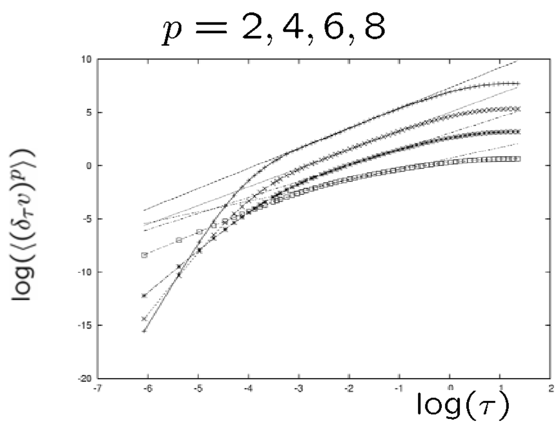
t

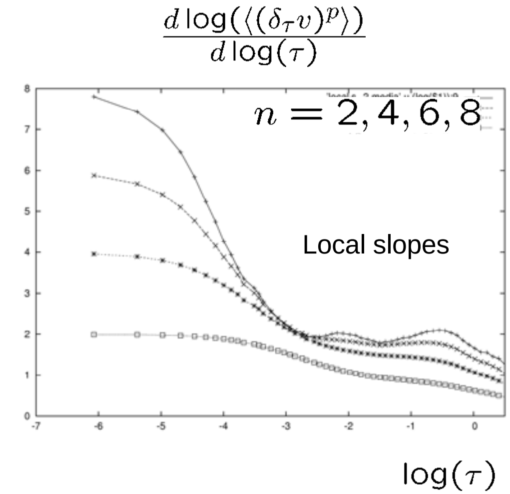
$\mathcal{P}(\tilde{a})$

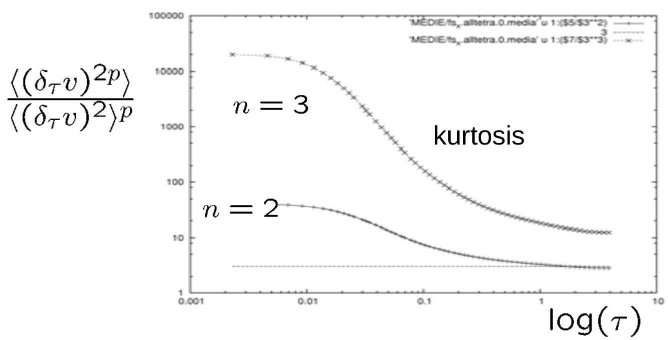


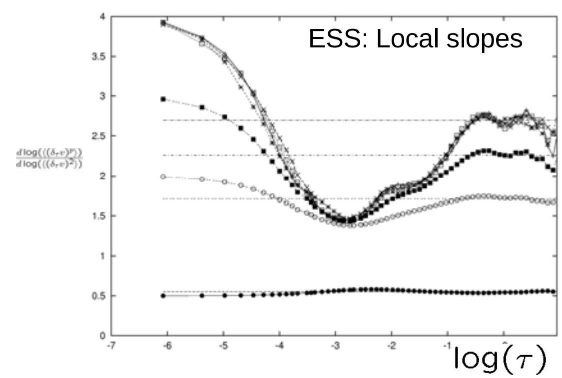
$\tilde{a} = a/\sigma_a$

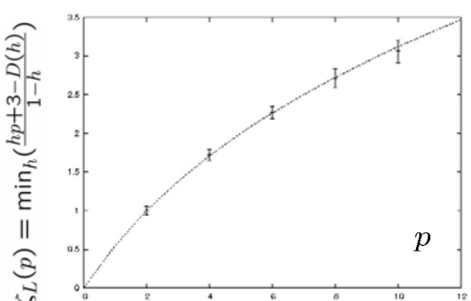
Single particle statistics

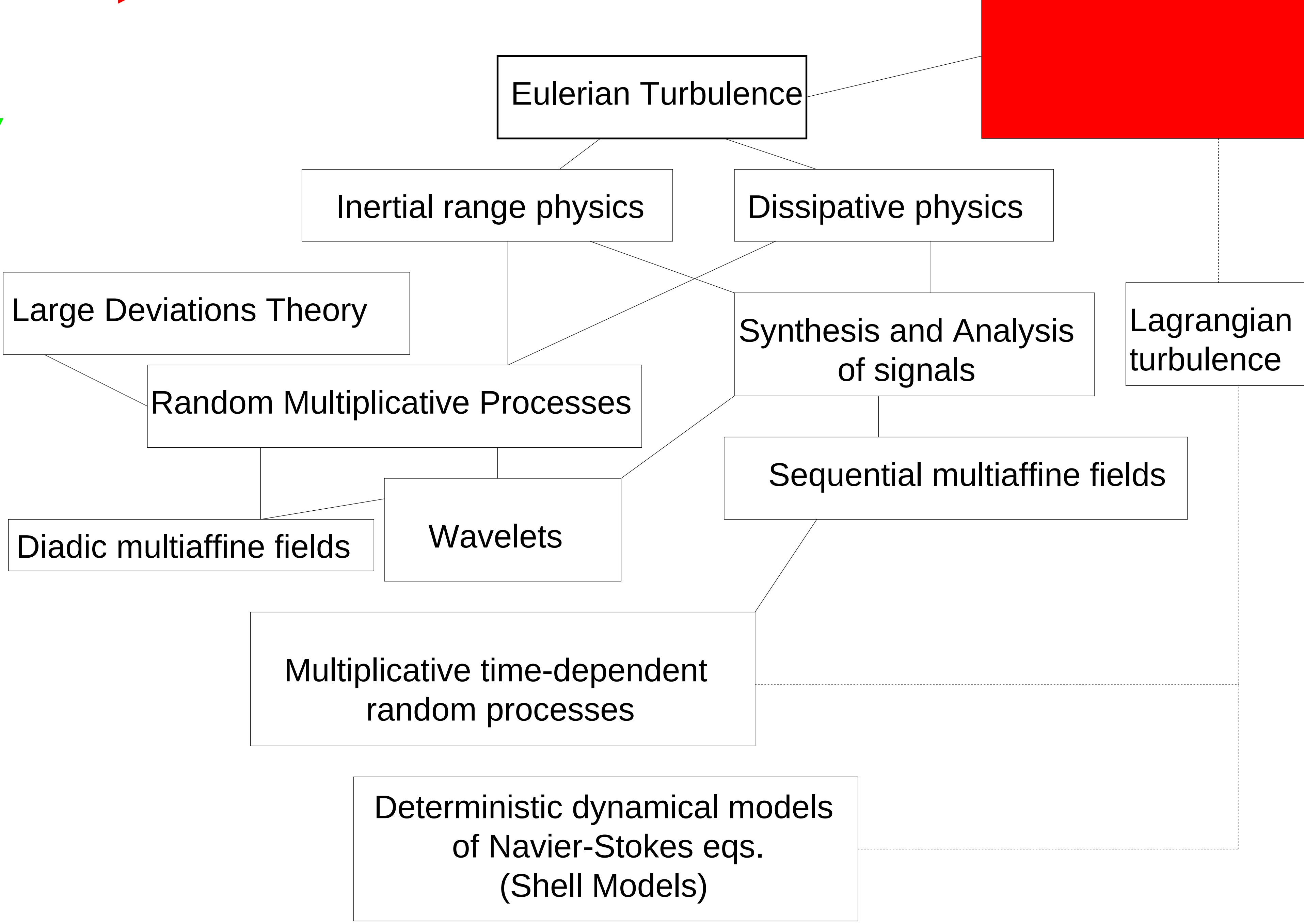











Eulerian Turbulence
Inertial range physics
Dissipative physics
Large Deviations Theory
Synthesis and Analysis of signals
Lagrangian turbulence
Random Multiplicative Processes
Sequential multiaffine fields
Diadic multiaffine fields
Wavelets
Multiplicative time-dependent random processes
Deterministic dynamical models of Navier-Stokes eqs. (Shell Models)

Personal view on “Modern issues in turbulence and scaling”

Multi-time multi-scale correlation functions:

$$S_n^{\bar{\alpha}}(\bar{\boldsymbol{r}}, \bar{t}) = \langle \delta_{\boldsymbol{r}_1} v^{\alpha_1}(t_1) \cdots \delta_{\boldsymbol{r}_n} v^{\alpha_n}(t_n) \rangle$$

Synthesis with the correct properties? Wavelets?
Analysis considering different geometrical configuration
connections with NS eqs. ?

Shell Models of Energy Cascade in Turbulence. L. Biferale *Ann. Rev. Fluid. Mech.* **35**, 441, 2003

Inverse structure functions, i.e. exit time statistics

$$\langle R(\delta v)^p \rangle \sim (\delta v)^{\chi(p)}$$

A way to characterize “laminar velocity fluctuations”:
2d turbulence,
2-particles diffusion,
Pick of velocity PDF in FDT

Inverse Statistics in two dimensional turbulence L. Biferale, M. Cencini, A. Lanotte and D. Vergni *Phys. Fluids* **15** 1012, 2003.

Sub-leading correction to scaling: anisotropy, non-homogeneity …

Are the corrections universal?
Quantify the leaading/sub-leading ratios
Phenomenology of the anisotropic fluctuations: is there a cascade?
Connection to NS eqs.

Anisotropy in Turbulent Flows and in Turbulent Transport L. Biferale and I. Procaccia . **nlin.CD/0404014**

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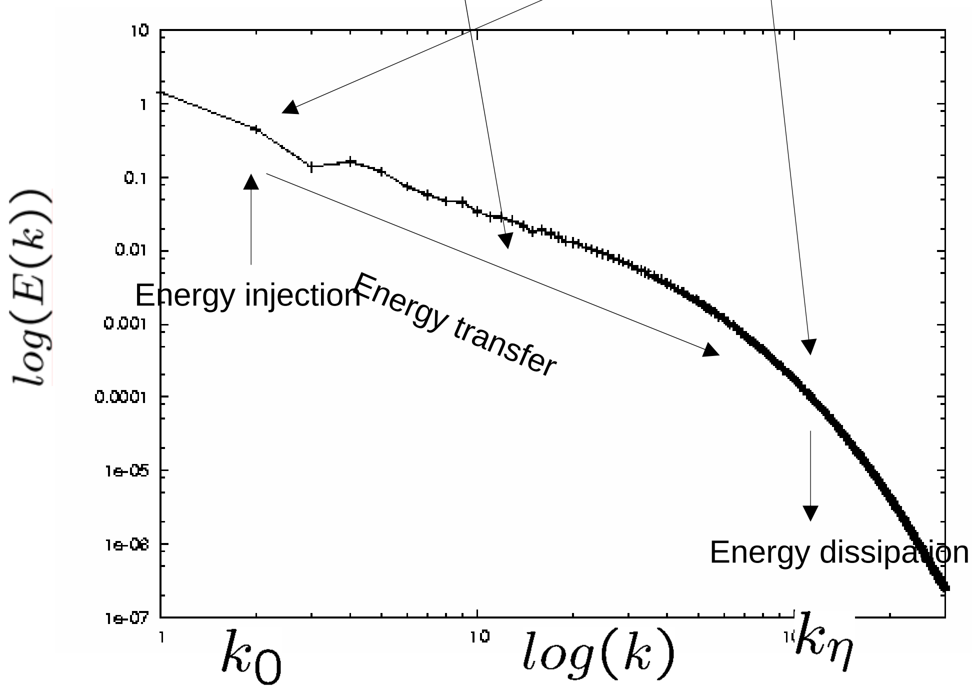
$$\partial_t \hat{\boldsymbol{v}}(\boldsymbol{k}) + \boldsymbol{k} \sum_{\boldsymbol{p},\boldsymbol{q}} \hat{\boldsymbol{v}}(\boldsymbol{p})\hat{\boldsymbol{v}}(\boldsymbol{q}) = \nu k^2 \hat{\boldsymbol{v}}(\boldsymbol{k}) + \hat{\boldsymbol{f}}(\boldsymbol{k})$$



$$E(k) = \int_{\boldsymbol{k}=k} d\boldsymbol{k} \langle |\boldsymbol{u}(\boldsymbol{k})|^2 \rangle$$

$$k_0 < k < k_\eta$$

Inertial range of scales